B.A./B.Sc. 3rd Semester (Honours) Examination, 2022 (CBCS)

Subject : Mathematics

Course : CC-VI (BMH3CC06)

(Group Theory-I)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

(Notations and symbols have their usual meaning.)

1. Answer *any ten* questions: 2×10=20

(a) For any integer $n > 2$. Show that there are at least two elements in $U(n)$ that satisfy $x^2 = 1$.

(b) Let G be a group with property that for any x, y, z in the group G, $xy = zx$ implies $y = z$. Prove that G is an Abelian.

(c) List the six elements of $GL(2, \mathbb{Z}_2)$. Show that the group is non-Abelian by finding two elements that do not commute.

(d) Find the order of f, where

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 5 & 8 & 7 & 3 & 1 & 9 & 2 & 4 & 6 \end{pmatrix}.$$

(e) Let $o(G) = 8$. Show that G must have an element of order 2.

(f) Find the inverse of the element $\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ in $GL(2, \mathbb{Z}_{11})$.

(g) If $(\mathbb{R}^*, \cdot)$ is the group of non-zero real numbers under multiplication, then show that $(\mathbb{R}^*, \cdot)$ is not isomorphic to $(\mathbb{R}, +)$, the group of real numbers under addition.

(h) Give an example of a group G and elements $a, b \in G$ such that $o(a)$ and $o(b)$ are finite but ab is not of finite order.

(i) If G has only one element of order n, then show that $a \in Z(G)$.

(j) Let m and d be two natural numbers, then show that $m\mathbb{Z} \subseteq d\mathbb{Z}$ iff d divides m.

(k) Prove that $(Q+)$ is not cyclic group.

(l) Let $f: GL(n, \mathbb{R}) \to \mathbb{R}^*$ be defined by $f(A) = \det A$. Then show that f is homomorphism.

(m) Let G be a group and $x \neq e$ be an element in G. Then show that there exists a unique homomorphism $f: \mathbb{Z} \to G$ such that $f(1) = x$.

(n) Prove that ${}^{8\mathbb{Z}}/_{56\mathbb{Z}} \simeq \mathbb{Z}_7$.

(o) If H is a normal sub-group of G. Then show that $f(H)$ need not be normal in G.

5×4=20

2. Answer *any four* questions:

(a) (i) Suppose that a and b are group elements that commute and have order m and n respectively. If $(a) \cap (b) = \{e\}$, then prove that the group contains an element whose order is the least common multiple of m and n.

(ii) If p is a prime number, then prove that $\mathbb{Z}_p$ is a cyclic group. 3+2=5

(b) Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a, b \in \mathbb{R},\ a > 0 \right\}$ is a group under matrix multiplication and let $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$ be subset of G. Show that H is a normal subgroup of G and $G/_H \cong \mathbb{R}$. 2+3=5

(c) Let G be a group of order $2p$ where p is prime number greater than 2. Prove that G is isomorphic to $\mathbb{Z}_{2p}$ or D_p.

(d) (i) Let $H = \{z \in \mathbb{C}^* : |z| = 1\}$. Prove that $\mathbb{C}^*/_H$ is isomorphic to $\mathbb{R}^+$, the group of positive real numbers under multiplication.

(ii) Suppose G is a finite group of order n and m is relatively prime to n. If $g \in G$ and $g^m = e$, then prove that $g = e$. 3+2=5

(e) If G is a finite group and H is a sub-group of G, then prove that $o(H) | o(G)$. Is the converse of the above statement true? Justify your answer. 3+2=5

(f) Let H and K be two sub-groups of a group G. Prove that HK is a sub-group of G if and only if $HK = KH$.

10×2=20

3. Answer *any two* questions:

(a) (i) State and prove Cauchy's theorem for finite abelian groups.

(ii) Determine all homomorphism from $\mathbb{Z}$ onto S_3. (1+4)+5=10

(b) (i) If a group G has finite number of sub-groups, then show that G is finite group.

(ii) Let A, B be finite cyclic groups of order m and n respectively. Prove that $A \times B$ is cyclic if and only if m and n are relatively prime. 5+5=10

(c) (i) Let G be an non-abelian group of order pq where p, q are prime numbers, then prove that $o\big(Z(G)\big) = 1$.

(ii) Show that a group of order 4 is either cyclic or is an direct product of two cyclic groups of order 2 each. 5+5=10

(d) (i) Prove that a finite group of order n is isomorphic to a sub-group of S_n.

(ii) Let G be a finite commutative group of order n and $\gcd(m, n) = 1$. Prove that $\phi : G \to G$ defined by $\phi(x) = x^m$, $x \in G$ is an isomorphism.

(iii) Let G_1 and G_2 be two groups. If N_1 and N_2 are two normal sub-groups of G_1 and G_2 respectively, then prove that $N_1 \times N_2$ is a normal sub-group of $G_1 \times G_2$ and $\frac{G_1 \times G_2}{N_1 \times N_2} \simeq \frac{G_1}{N_1} \times \frac{G_2}{N_2}$. 3+3+4=10